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or,

$$J(\bar{C}_{12}) > \frac{r_1^3 + r_3^3}{3} + J(\bar{C}_{32}).$$

Now

$$J(\bar{C}_{32}) = \int_3^2 r^2 \sqrt{r'^2 + r^2\,\theta'^2}\, dt \geqq \int_3^2 r^2 |r'|\, dt,$$

$$J(\bar{C}_{32}) \geqq \left| \int_3^2 r^2 r' dt \right| = \frac{|r_2^3 - r_3^3|}{3} \geqq \frac{r_2^3 - r_3^3}{3}.$$

Hence

$$J(\bar{C}_{12}) > \frac{r_1^3 + r_2^3}{3} = J(S_{102}).$$

We may therefore state the theorem:

THEOREM III. *If* $|\theta_2 - \theta_1| \geqq \pi/3$ *the curve* S_{102}, *formed by the straight lines* $\theta = \theta_1$, $\theta = \theta_2$, *has the least moment of inertia with respect to the point* 0 *of all curves connecting the points* 1 *and* 2.

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TRIANGLES AND QUADRILATERALS INSCRIBED TO A CUBIC AND CIRCUMSCRIBED TO A CONIC

BY H. S. WHITE

PONCELET'S porism of inscribed and circumscribed polygons has a certain extension to the system of a non-singular plane cubic and a curve of the second class, resembling that recently given by Morley to point cubic and line cubic, but differing also in one feature. I shall consider only polygons of three and of four sides, first as to their possible occurrence, then by reviewing a well known construction, and lastly as to the corresponding algebraic conditions.

1. Can a triangle have vertices moving freely on a cubic while its sides continually touch a conic? If not on a general cubic, then can a special cubic admit such a system of triangles, and how strongly must it be specialized?

Call the cubic C and the conic K, and suppose that every point of C is a vertex of a triangle inscribed in C and having for sides three tangents of K. If one side of such a triangle is a, and two vertices upon that side are the points A_1 and A_2 of the cubic, let b denote a second side, and B_1, B_2, its vertices, so that A_2 and B_1 shall denote the same point. Outside of these points, a and b cut the cubic in A_3 and B_3 respectively, and these points by hypothesis must be themselves vertices of other such triangles. Through A_3, however, pass only 2 tangents of K; one is a, call the other one d. Of this second triangle, two vertices must lie on a; one is A_3, let the other be $A_2 (\equiv B_1)$. Then as a and b are the two tangents to K from A_2, both must belong to this second triangle, and B_3 must be its third vertex. Similarly we should find a third triangle B_3, $B_2 (\equiv C_1)$, C_3, and a fourth $C_3, C_2 (\equiv A_1), A_3$. Hence *the existence of a triangle with vertices freely moving on the cubic while its sides roll upon the conic necessitates the existence of a quadrilateral with all six vertices moving freely on the cubic while its sides continually touch the conic.*

It is well known that if a complete quadrilateral has all its vertices upon the cubic, its opposite vertices are always a Steiner pair of the lowest kind, and that every cubic contains three discrete systems of such pairs. To construct the quadrilateral, two pairs may be chosen at random from any one system, then the lines joining points of one pair to points of the other constitute the quadrilateral, and its remaining vertices will fall on a third pair of the same system. Accordingly *if the desired quadrilateral can exist, its pairs of opposite vertices must be three conjugate pairs in the same one of the three systems in involution on the cubic.*

From every point of the cubic C, the pairs of points in involution on C are projected by the pairs of rays of a pencil in involution. Two rays u_1 and u_2 of each such pencil are tangent to K; replace them by their corresponding rays in the involution, v_1 and v_2. In how many pencils will it happen that v_1 coincides with u_2; i. e., that the two tangents of K from a point Z of the cubic are paired in the involution on the cubic? To answer this question, examine the whole system of lines v_1, v_2, or the v's, and determine its class.

Through Z there pass two v's conjugate to the two tangents from Z to K. But also through Z', the conjugate of Z on C, there pass two tangents of K which belong to pencils emanating from four other points; the four rays conjugate to these tangents will be additional v-lines through Z, consequently the system of v's is of class 6, and so contains $6 \times 2 = 12$ tangents of K. Some of those 12 however are improperly counted, namely those which are self-conjugate in the involutions. Self-conjugate lines (call them w-lines) fill a locus of class 3, known in certain relations as a Cayleyan envelope. Hence $3 \times 2 = 6$ of them are tangent to K, leaving but 6 tangent v-lines properly so called. As these 6 are at once u-lines and v-lines, they occur in pairs, and give only 3 points of meeting of such pairs on the cubic.

If the Steiner correspondence or involution is given, together with the cubic and conic, these 3 points are susceptible of algebraic determination, and we can infer that if any cubic contains more than 3 such points, every point upon it will satisfy the same conditions. So it is proved that

Every given cubic contains 3 points from which pass pairs of tangents to a given conic, each pair consisting of lines conjugate in one involution on the cubic. As the cubic has in general three such involutions, it will have altogether 9 such points. Further, if in one of the involutions there occur 4 such points, then every point of the cubic (except its intersections with the conic) will be the intersection of 2 conjugate tangents to the conic.

When this special case occurs, it is not difficult to show that quadrilaterals are present, of the sort described above. As one quadrilateral has six vertices, we see that the existence of one quadrilateral inscribed in the cubic and circumscribed to the conic is more than sufficient to prove the existence of an infinite system of quadrilaterals having the same property.

2. If an infinite system of Poncelet quadrilaterals is present, consider the cubic as hessian of a primitive cubic C_1; then the three diagonals of each quadrilateral will be tangents to the Cayleyan of C_1, and form a polar triangle of the conic, variable upon the Cayleyan. This gives at once a means of writing explicitly a relation between the conic and cubic, namely by using the condition that a given curve of the third class shall have tangents arranged in an infinite number of triples, each triple a polar triangle of the conic. That condition consists of the identical vanishing of a quadric covariant of the Cayleyan and the conic.*

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* See *Transactions of the American Mathematical Society*, vol. 4 (1903), p. 138, (5).

As that covariant condition is incorrectly printed in its first publication,* it is desirable to give it here *in extenso*. Call the equation of the Cayleyan $u_k^3 \equiv u_{k'}^3 = 0$, and that of the conic $u_\alpha^2 = u_\beta^2 = \text{etc.} = 0$, then the condition is

$$\begin{aligned}\Gamma'(u) \equiv \Delta^2(\alpha k k')^2 u_k\, u_{k'} &+ 3\,\Delta\,(\alpha\beta k)(\alpha\beta k')(\gamma\delta k)^2 u_{k'}^2 \\ &- 3(\alpha\beta k)(\alpha\beta k')(\gamma\delta k)^2(\epsilon\zeta k')^2 .\ u_\theta^2 \\ &- \Delta\,(\alpha\beta k)(\alpha\beta k')(\gamma k k')^2\, u_\delta^2 \equiv 0,\end{aligned}$$

where Δ denotes the discriminant $(\alpha\beta\gamma)^2$. This identity in variables (u) yields however but two independent conditions. These are necessary, but of course not sufficient, since we have seen that only a double infinity of conics can be found, and accordingly *three* conditions are to be imposed. Such a set is found in the following manner.

It is well known that in a doubly infinite linear system (or web) of line-conics, four common tangents to any two of the conics will intersect in six points of a fixed point-cubic. The pairs of opposite points in any such quadrilateral are Steiner pairs of the lowest kind upon the point-cubic. That cubic is Cayleyan to a certain line-cubic $\Pi(u) = 0$, whose system of quadric polars consists of the line-conics in the web first mentioned; and if the point-cubic be given, the line-cubic $\Pi(u) = 0$ is one of three determinate curves, so that we find three doubly infinite webs of conics standing in the required relation. Probably the simplest statement is obtained by assuming as known a point-cubic, $f = 0$, of which the given cubic, $\Delta^3(x) = 0$, is hessian. The form $f(x)$ determines uniquely a doubly apolar form $\Pi(u)$, whose first polars are then the conics about which quadrilaterals can be circumscribed so as to move freely while inscribed to the hessian. Analytically the conditions can be written, first in terms of a single one of the three pro-hessians, $f(x) = 0 = f_x^3$, of the given cubic $\Delta_x^3 = 0$.

$$\begin{aligned}&1)\quad (\Delta\Delta' u)^2\Delta_\phi\Delta'_\phi - \tfrac{1}{6}\, S(ff'u)^2 f_\phi f'_\phi \equiv \mu \cdot u_\phi^2, \\ &2)\quad (\Delta\Delta' u)^2\Delta_\psi\Delta'_\psi - \tfrac{1}{6}\, S(ff'u)^2 f_\psi f'_\psi \equiv -\mu \cdot u_\psi^2, \\ &3)\quad f_\chi^2 f_y \equiv 0.\end{aligned}$$

In these defining equations, (1) and (2) are identities in variables (u), S is the invariant of degree 4 in the coefficients of f_x^3, μ and $-\mu$ are constants whose value depends on the cubic f_x^3 alone,* and $u_\phi^2 = 0$, $u_\psi^2 = 0$, $u_\chi^2 = 0$ denote the conics of the three systems respectively; the identity (3) is in variables (y). These equations have the merit of being linear in the coefficients of the conic. All three systems may be defined by equations like (3) if we assume as known the three pro-hessians $f_x^3 = 0$, $g_x^3 = 0$, and $h_x^3 = 0$ of the given cubic $\Delta_x^3 = 0$. The conditions are then:

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* The explicit reckonings following p. 138, loc. cit., are correct, the only errors being in formulæ (4) and (5).

$$f_\chi^2 f_y \equiv 0,\ g_\phi^2 g_y \equiv 0, \text{ and } h_\psi^2 h_y \equiv 0.$$

These are well known systems, but this their peculiar property seems not to have been exhibited as an extension of the theorems connected with Poncelet's polygons.† Schroeter's celebrated linear construction of conjugate point pairs on the cubic from three given pairs requires the drawing of exactly the lines which have been described as tangents to conics, four at a time giving rise to one new pair of vertices. Calling these a Schroeter quadrilateral, we may formulate what has been proved:

A cubic curve being determined by Schroeter's construction from three pairs of points, the eight sides of any two Schroeter quadrilaterals are tangent to some conic, which envelopes a simply infinite system of such quadrilaterals; and all such conics form a doubly infinite linear system of line-conics, the first polars of a line-cubic which has Schroeter's cubic for its Cayleyan.

Given one Schroeter quadrilateral and two conjugate points not upon its sides, another such quadrilateral can be constructed by solving a single problem of the second order, or a quadratic equation.

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* See Conics and Cubics connected with a plane cubic etc., *Transactions of the Amer. Math. Society*, vol. 1 (1900), p. 8.

† See however Emil Weyr, Uber Involutionen höherer Grade, *Crelle's Journal*, vol. 72 (1870), pp. 285–292, and S. Kantor, Verallgemeinerungen eines Poncelet'schen Satzes, *Crelle's Journal*, vol. 86 (1879), p. 271.